Household Problem

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Abstract

This is a simple household problem in a one period economy.

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1 Household Problem with 2 Goods

Household preferences are gives as

$$u(c_1, c_2) = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma}$$

The budget constraint is

$$p_1 c_1 + p_2 c_2 = I,$$

where income I is exogenously given (endowment income).

1.1 Budget Constraint

If you want to draw the budget constraint you can reformulate the budget constraint as

$$c_2 = \overbrace{\frac{I}{p_2}}^{\text{Intercept}} \overbrace{-\frac{p_1}{p_2}}^{\text{slope}} c_1.$$

The extreme consumption points, where the budget constraint crosses the axes are:

$$c_1 = 0 \rightarrow c_2 = \frac{I}{p_2},$$
$$c_2 = 0 \rightarrow c_1 = \frac{I}{p_1}.$$

Draw this budget constraint with quantities of c_1 on the horizontal axis and quantities of c_2 on the vertical axis.

1.2 Household Maximization Problem

The household maximization problem is:

$$max_{\{c_1, c_2\}} \left\{ \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma} \right\}$$
$$s.t.$$
$$p_1 c_1 + p_2 c_2 = I.$$

The optimality condition of the household is

$$MRS \equiv \frac{MU_{c_1}}{MU_{c_2}} = \frac{p_1}{p_2},$$

where $MU_{c_1} = \frac{\partial u(c_1,c_2)}{\partial c_1}$ is the marginal utility w.r.t. c_1 and $MU_{c_2} = \frac{\partial u(c_1,c_2)}{\partial c_2}$. With the functional form given above the MRS becomes:

$$MU_{c_1} = \frac{\partial u(c_1, c_2)}{\partial c_1} = (1-\sigma)\frac{c_1^{1-\sigma-1}}{(1-\sigma)} = c_1^{-\sigma},$$
$$MU_{c_2} = \frac{\partial u(c_1, c_2)}{\partial c_2} = (1-\sigma)\frac{c_2^{1-\sigma-1}}{(1-\sigma)} = c_2^{-\sigma},$$

and

$$MRS = \frac{c_1^{-\sigma}}{c_2^{-\sigma}} = \frac{c_2^{\sigma}}{c_1^{\sigma}} = \left(\frac{c_2}{c_1}\right)^{\sigma}.$$

The optimality condition says that this has to be equal to the price ratio so that

$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \frac{p_1}{p_2}.$$

We can now express c_2 as a function of $c_1$and prices

$$\begin{aligned}
&\left(\frac{c_2}{c_1}\right)^{\sigma} = \frac{p_1}{p_2}, \\
&\quad \to \left(\left(\frac{c_2}{c_1}\right)^{\sigma}\right)^{\frac{1}{\sigma}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}}, \\
&\quad \to \left(\frac{c_2}{c_1}\right)^{\sigma\times\frac{1}{\sigma}} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}}, \\
&\quad \to \frac{c_2}{c_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}}, \\
&\quad \to c_2 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}} c_1.
\end{aligned}$$

You can use this equation together with the household budget constraint to solve for $c_1$and c_2 because p_1, p_2, I and σ will all be given:

$$p_1 c_1 + p_2 c_2 = I, \tag{1}$$
$$c_2 = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}} c_1. \tag{2}$$

Two equations in two unknowns. Solve it for $c_1^*$and c_2^*. Plug the second into the first equation

$$
\begin{aligned}
p_1 c_1 + p_2 \overbrace{\left[\left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}} c_1\right]}^{c_2} &= I \\
\rightarrow p_1 c_1 + p_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}} c_1 &= I \\
\rightarrow \left(p_1 + p_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}}\right) c_1 &= I \\
\rightarrow c_1^* &= \frac{I}{\left(p_1 + p_2 \left(\frac{p_1}{p_2}\right)^{\frac{1}{\sigma}}\right)}.
\end{aligned}
$$

The solve for c_2^* using equation (2)

2 Household Problem with 2 Goods and Taxes

Household preferences are gives as

$$
u(c_1, c_2) = \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma}
$$

The budget constraint is

$$
p_1 c_1 + \overbrace{(1+\tau) \times p_2}^{\bar{p}_2} c_2 = I - T,
$$

where τ is a proportional tax on good 2 and T is a lump sum tax and income I is exogenously given (endowment income). You could simply treat this as good 2 having a new, higher price now that is defined as

$$
\bar{p}_2 := (1+\tau) p_2
$$

and solve everything with this new price. In these notes I'll keep it separate so that the tax rate appears explicitly. Wherever you had p_2 before you know have $(1+\tau) p_2$.

2.1 Budget Constraint

If you want to draw the budget constraint you can reformulate the budget constraint as

$$
c_2 = \overbrace{\frac{I-T}{(1+\tau) p_2}}^{\text{Intercept}} - \overbrace{\frac{p_1}{(1+\tau) p_2}}^{\text{slope}} c_1.
$$

The extreme consumption points, where the budget constraint crosses the axes are:

$$c_1 = 0 \rightarrow c_2 = \frac{I}{(1+\tau)\, p_2},$$
$$c_2 = 0 \rightarrow c_1 = \frac{I}{p_1}.$$

Draw this budget constraint with quantities of c_1 on the horizontal axis and quantities of c_2 on the vertical axis.

2.2 Household Maximization Problem

The household maximization problem is:

$$max_{\{c_1,c_2\}} \left\{ \frac{c_1^{1-\sigma}}{1-\sigma} + \frac{c_2^{1-\sigma}}{1-\sigma} \right\}$$
$$s.t.$$
$$p_1 c_1 + (1+\tau)\, p_2 c_2 = I - T.$$

The optimality condition of the household is

$$MRS \equiv \frac{MU_{c_1}}{MU_{c_2}} = \frac{p_1}{(1+\tau)\, p_2},$$

where $MU_{c_1} = \frac{\partial u(c_1,c_2)}{\partial c_1}$is the marginal utility w.r.t. c_1 and $MU_{c_2} = \frac{\partial u(c_1,c_2)}{\partial c_2}$. With the functional form given above the MRS becomes:

$$MU_{c_1} = \frac{\partial u\,(c_1, c_2)}{\partial c_1} = (1-\sigma)\, \frac{c_1^{1-\sigma-1}}{(1-\sigma)} = c_1^{-\sigma},$$
$$MU_{c_2} = \frac{\partial u\,(c_1, c_2)}{\partial c_2} = (1-\sigma)\, \frac{c_2^{1-\sigma-1}}{(1-\sigma)} = c_2^{-\sigma},$$

and

$$MRS = \frac{c_1^{-\sigma}}{c_2^{-\sigma}} = \frac{c_2^{\sigma}}{c_1^{\sigma}} = \left(\frac{c_2}{c_1}\right)^{\sigma}.$$

The optimality condition says that this has to be equal to the price ratio so that

$$\left(\frac{c_2}{c_1}\right)^{\sigma} = \frac{p_1}{(1+\tau)\, p_2}.$$

We can now express c_2 as a function of $c_1$and prices

$$\begin{aligned}
\left(\frac{c_2}{c_1}\right)^{\sigma} &= \frac{p_1}{(1+\tau)\,p_2}, \\
\rightarrow \left(\left(\frac{c_2}{c_1}\right)^{\sigma}\right)^{\frac{1}{\sigma}} &= \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}}, \\
\rightarrow \left(\frac{c_2}{c_1}\right)^{\sigma\times\frac{1}{\sigma}} &= \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}}, \\
\rightarrow \frac{c_2}{c_1} &= \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}}, \\
\rightarrow c_2 &= \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}} c_1.
\end{aligned}$$

You can use this equation together with the household budget constraint to solve for $c_1$and c_2 because p_1, p_2, I and σ will all be given:

$$p_1 c_1 + (1+\tau)\,p_2 c_2 = I - T, \tag{3}$$

$$c_2 = \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}} c_1. \tag{4}$$

Two equations in two unknowns. Solve it for $c_1^*$and c_2^*. Plug the second into the first equation

$$\begin{aligned}
p_1 c_1 + (1+\tau)\,p_2 \overbrace{\left[\left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}} c_1\right]}^{c_2} &= I - T \\
\rightarrow p_1 c_1 + (1+\tau)\,p_2 \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}} c_1 &= I - T \\
\rightarrow \left(p_1 + (1+\tau)\,p_2 \left(\frac{p_1}{(1+\tau)\,p_2}\right)^{\frac{1}{\sigma}}\right) c_1 &= I - T \\
\rightarrow c_1^* &= \frac{I - T}{\left(p_1 + (1+\tau)\,p_2 \left(\frac{p_1}{(1+\tau)p_2}\right)^{\frac{1}{\sigma}}\right)}.
\end{aligned}$$

The solve for c_2^* using equation (4).

3 Household Problem with Consumption and Leisure and Taxes

Household preferences are gives as

$$u(c,\ell) = \frac{c^{1-\sigma}}{1-\sigma} + \frac{\ell^{1-\sigma}}{1-\sigma}$$

The budget constraint is

$$c = \overbrace{(1-\tau)\times(h-\ell)\times w}^{\text{after tax income}} - T$$

where τ is a proportional tax on labor income and T is a lump sum tax. We could reformulate this as

$$p_1 c + \overbrace{(1-\tau)\times w}^{\bar{p}_2}\ell = \overbrace{(1-\tau)\,hw}^{I} - T,$$

in which case it would look identical to the earlier problem. Note that $(1-\tau)\,w$ is the price of leisure. However, I will keep it as is and solve with the new notation.

3.1 Budget Constraint

If you want to draw the budget constraint you can reformulate the budget constraint as

$$c = \overbrace{(1-\tau)\,wh - T}^{\text{intercept}} - \overbrace{(1-\tau)\,w}^{\text{slope}} \times \ell.$$

The extreme consumption points, where the budget constraint crosses the axes are:

$$c = 0 \rightarrow \ell = \frac{(1-\tau)\,wh - T}{(1-\tau)\,w},$$
$$\ell = 0 \rightarrow c = (1-\tau)\,wh - T.$$

Draw this budget constraint with quantities of c on the horizontal axis and quantities of ℓ on the vertical axis.

3.2 Household Maximization Problem

The household maximization problem is:

$$\max_{\{c,\ell\}} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \frac{\ell^{1-\sigma}}{1-\sigma} \right\}$$
$$s.t.$$
$$c = (1-\tau)\,(h-\ell)\,w - T.$$

The optimality condition of the household is MRS equal to the price ratio or:

$$MRS \equiv \frac{MU_c}{MU_l} = \frac{1}{(1-\tau)\,w},$$

where $MU_c = \frac{\partial u(c,\ell)}{\partial c}$ is the marginal utility w.r.t. c and $MU_\ell = \frac{\partial u(c,\ell)}{\partial \ell}$. With the functional form given above the MRS becomes:

$$
\begin{aligned}
MU_c &= \frac{\partial u(c,\ell)}{\partial c} = (1-\sigma)\frac{c_1^{1-\sigma-1}}{(1-\sigma)} = c^{-\sigma}, \\
MU_\ell &= \frac{\partial u(c,\ell)}{\partial \ell} = (1-\sigma)\frac{\ell^{1-\sigma-1}}{(1-\sigma)} = \ell^{-\sigma},
\end{aligned}
$$

and

$$
MRS = \frac{MU_c}{MU_\ell} = \frac{c^{-\sigma}}{\ell^{-\sigma}} = \frac{\ell^\sigma}{c_1^\sigma} = \left(\frac{\ell}{c}\right)^\sigma .
$$

The optimality condition says that this has to be equal to the price ratio so that

$$
\left(\frac{\ell}{c}\right)^\sigma = \frac{1}{(1-\tau)w}.
$$

We can now express ℓ as a function of $c$and prices

$$
\begin{aligned}
\left(\frac{\ell}{c}\right)^\sigma &= \frac{1}{(1-\tau)w}, \\
\rightarrow \left(\left(\frac{\ell}{c}\right)^\sigma\right)^{\frac{1}{\sigma}} &= \left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}}, \\
\rightarrow \left(\frac{\ell}{c}\right)^{\sigma\times\frac{1}{\sigma}} &= \left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}}, \\
\rightarrow \frac{\ell}{c} &= \left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}}, \\
\rightarrow \ell &= \left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}} c.
\end{aligned}
$$

You can use this equation together with the household budget constraint to solve for $c$and ℓ because w, h and σ will all be given:

$$
c + (1-\tau)w\ell = (1-\tau)hw - T, \tag{5}
$$

$$
\ell = \left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}} c. \tag{6}
$$

Two equations in two unknowns. Solve it for $c^*$and ℓ^*. Plug the second into the first equation

$$
\begin{aligned}
c+(1-\tau)\,w\overbrace{\left[\left(\frac{1}{(1-\tau)\,w}\right)^{\frac{1}{\sigma}}c\right]}^{\ell} &= (1-\tau)\,hw-T \\
\to c+(1-\tau)\,w\left(\frac{1}{(1-\tau)\,w}\right)^{\frac{1}{\sigma}}c &= (1-\tau)\,hw-T \\
\to \left(1+(1-\tau)\,w\left(\frac{1}{(1-\tau)\,w}\right)^{\frac{1}{\sigma}}\right)c &= (1-\tau)\,hw-T \\
\to c_1^* &= \frac{(1-\tau)\,hw-T}{\left(1+(1-\tau)\,w\left(\frac{1}{(1-\tau)w}\right)^{\frac{1}{\sigma}}\right)}.
\end{aligned}
$$

The solve for ℓ using equation (6).

4 References

References