Dynamic General Equilibrium in 2-Period Model

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1 Introduction

This solution key solves a simplified version of the model in Chapter 11 of Williamson (2013). We shut down the labor labor–leisure choice in both periods and assume an inelastic labor supply. It would be a vertical line in the labor market demand–supply graph. This reduces the optimality conditions from

$$\begin{aligned} MRS_{c_1,c_2} &= \frac{1}{\left(\frac{1}{1+r}\right)}, \\ MRS_{l_1,c_1} &= w_1, \\ MRS_{l_2,c_2} &= w_2, \\ MRS_{l_1,l_2} &= \frac{w_1\,(1+r)}{w_2}, \end{aligned}$$

to just

$$MRS_{c_1,c_2} = \frac{1}{\left(\frac{1}{1+r}\right)}, \tag{1}$$

a much simpler problem to solve. The other change that we made was to assume that the capital stock is determined by the households and not the firms. Household savings determines the supply of capital.

2 Setup

An individual lives for two periods. There are N_1 young households in the economy and N_2 old households. Preferences of an individual household are given as

$$u(c_1, c_2) = \ln(c_1) + \beta \times \ln(c_2),$$

where c_1 is consumption today and c_2 is consumption tomorrow, and β is the time preference factor. The young household supplies one unit of labor inelastically and earns the wage rate w for it in the first period. In addition, the household pays a proportional tax τ on labor in the first period. The household can save in the first period and earn interest r when entering the second period. In the second period the household retires and has no further labor income. The only income the household receives is from savings.

There is a representative firm that produces output using the following production function: $F(K, L) = A \times \left(K^{\alpha} + L^{1-\alpha}\right)$, where A is the total factor productivity, K is aggregate stock of capital, and L is aggregate labor. Assume that capital depreciates fully between the two periods so that $\delta = 1$ or 100 percent.

3 Solution

3.1 Preliminary Discussion

A 100 percent depreciation rate implies that the law of motion of capital

$$K' = (1-\delta)\,K + I$$

collapses to

$$K' = I.$$

Later in equilibrium (or steady state) we see that $K = K'$ and since this is a closed economy we also know that household savings will be equal to investment so that

$$S = I.$$

In other words, the capital stock K that is used in production by the firms is determined by the aggregate savings of all the households

$$K = I = S = s \times N_1$$

since only the young households save. This will be an equilibrium conditions and will appear in Section 3.4.

3.2 Households

First we solve the household problem:

$$\max_{\{c_1, c_2, s\}} \{\ln(c_1) + \beta \times \ln(c_2)\} \text{ s.t.} \tag{2}$$

$$c_1 + s = w(1-\tau), \tag{3}$$

$$c_2 = (1+r) \times s. \tag{4}$$

There are two ways to solve the household problem, they both result in the same solution.

Solution Method 1: Substitution. Substitute the budget constraint into the preferences so that you are left with a simple maximization over savings s. You then have

$$\max_s \{\ln(w(1-\tau) - s) + \beta \ln((1+r)s)\}.$$

We derive this w.r.t. s and get a FOC

$$\frac{1}{w(1-\tau) - s}(-1) + \frac{\beta}{(1+r)s}(1+r) = 0.$$

We can solve this for optimal savings[1]

$$s^* = \frac{\beta}{1+\beta}(1-\tau)w. \tag{5}$$

You can substitute this back into the household budget constraints (3) and (4) to get expressions for optimal consumption in the first and second period as[2]

$$c_1^* = \left(\frac{1}{1+\beta}\right) w(1-\tau), \tag{6}$$

$$c_2^* = \left(\frac{\beta}{1+\beta}\right)(1+r)(1-\tau)w. \tag{7}$$

Solution Method 2: Lifetime Budget Constraint. We can consolidate the two budget constraint into a lifetime budget constraint by substituting out savings s and get

$$c_1 + \frac{1}{1+r}c_2 = w(1-\tau).$$

[1]

$$\begin{aligned}
&\rightarrow \frac{1}{w(1-\tau)-s} = \frac{\beta}{(1+r)s}(1+r),\\
&\rightarrow \frac{1}{w(1-\tau)-s} = \frac{\beta}{s},\\
&\rightarrow s = \beta(w(1-\tau)-s),\\
&\rightarrow s + \beta s = \beta w(1-\tau),\\
&\rightarrow s = \frac{\beta}{1+\beta}(1-\tau)w,
\end{aligned}$$

[2]

$$\begin{aligned}
&c_1^* = w(1-\tau) - s^*,\\
&\rightarrow c_1^* = w(1-\tau) - \frac{\beta}{1+\beta}(1-\tau)w,\\
&\rightarrow c_1^* = \left(1 - \frac{\beta}{1+\beta}\right) w(1-\tau),\\
&\rightarrow c_1^* = \left(\frac{1+\beta-\beta}{1+\beta}\right) w(1-\tau),\\
&\rightarrow c_1^* = \left(\frac{1}{1+\beta}\right) w(1-\tau),
\end{aligned}$$

and

$$\begin{aligned}
&c_2^* = (1+r) \times s^*,\\
&\rightarrow c_2^* = \left(\frac{\beta}{1+\beta}\right)(1+r)(1-\tau)w.
\end{aligned}$$

We can set up the Lagrangian or use the optimality conditions (MRS=price ratio) directly to get

$$L() = \ln(c_1)+\beta\times\ln(c_2)+\lambda\left(w(1-\tau)-c_1-\left(\frac{1}{1+r}\right)c_2\right)$$

where λ is the Lagrange multiplier. Deriving this expression w.r.t. c_1, c_2 and λ we get the following system of optimality (or First Order Conditions FOCs)

$$\begin{aligned} \frac{1}{c_1} &= \lambda, \\ \frac{\beta}{c_2} &= \lambda\left(\frac{1}{1+r}\right), \\ w(1-\tau)-c_1-\left(\frac{1}{1+r}\right)c_2 &= 0. \end{aligned} \tag{8}$$

Substituting the first two equations we get the familiar optimality condition (MRS=price ratio) that —compare expression (1) in the introduction— relates today's to tomorrow's consumption

$$\frac{\left(\frac{1}{c_1}\right)}{\left(\frac{\beta}{c_2}\right)} = \frac{1}{\left(\frac{1}{1+r}\right)}. \tag{9}$$

This together with the lifetime budget constraint will solve the household problem for c_1^* consumption demand.[3] You can then use this to solve for c_2^*. Finally you can use one of the original budget constraints (3) or (4) to solve for optimal savings s^*. You will get the same results an in expressions (6) ,(7) and (5) . This will become your capital supply in equilibrium as described in Section 3.4.

[3] First use the optimality condition (9) to get an expression for c_2 as

$$\begin{aligned} &\rightarrow \left(\frac{1}{c_1}\right)\left(\frac{1}{1+r}\right) = \left(\frac{\beta}{c_2}\right), \\ &\rightarrow c_2 = \beta(1+r)c_1, \end{aligned}$$

and substitute this into the lifetime budget constraint (8)

$$\begin{aligned} &w(1-\tau)-c_1-\left(\frac{1}{1+r}\right)\beta(1+r)c_1 = 0, \\ &\rightarrow c_1+\left(\frac{1}{1+r}\right)\beta(1+r)c_1 = w(1-\tau), \\ &\rightarrow c_1\left(1+\left(\frac{1}{1+r}\right)\beta(1+r)\right) = w(1-\tau), \\ &\rightarrow c_1 = \frac{w(1-\tau)}{\left(1+\left(\frac{1}{1+r}\right)\beta(1+r)\right)}, \\ &\rightarrow c_1 = \frac{w(1-\tau)}{(1+\beta)}, \\ &\rightarrow c_1 = \left(\frac{1}{1+\beta}\right)w(1-\tau). \end{aligned}$$

3.3 Firms

Firms solve the following profit maximization problem

$$\max_{\{K,L\}}\left\{A\times\left(K^\alpha+L^{1-\alpha}\right)-r\times K-w\times L\right\}, \tag{10}$$

so that the optimality conditions (or FOCs) become:

$$\partial K:\ \alpha A K^{\alpha-1} = r, \tag{11}$$

$$\partial L:\ (1-\alpha)AL^{-\alpha} = w. \tag{12}$$

Expression (11) is capital demand and expression (12) is labor demand.

3.4 Equilibrium

We now put the household and the firm sides together.

Definition 1. Given prices for labor w and capital r a competitive equilibrium is an allocation of $\{c_1, c_2, K, L\}$ such that
(i) the household optimization problem (9) is satisfied,
(ii) the firm's profit maximization problem (10) is satisfied
(iii) all markets clear, that is capital markets, labor

markets and goods markets clear as:

$$\begin{aligned}
K &= S = N_1 \times s^*, \\
L &= N_1 \times 1, \\
C &= N_1 \times c_1 + N_2 \times c_2, \\
C + S &= Y = A \times \left(K^\alpha + L^{1-\alpha}\right).
\end{aligned}$$

Given exogenous parameters $\{A, \beta, \alpha, \tau, N_1, N_2\}$ the solution (in equilibrium) for all 9 endogenous variables $\{c_1, c_2, s, C, K, L, Y, r, w\}$ is determined by the following system of 9 equations:

$$\begin{aligned}
c_1^* &= \left(\frac{1}{1+\beta}\right) w\left(1-\tau\right), \\
c_2^* &= \left(\frac{\beta}{1+\beta}\right) \left(1+r\right)\left(1-\tau\right) w, \\
s^* &= \frac{\beta}{1+\beta}\left(1-\tau\right) w, \\
C &= N_1 \times c_1^* + N_2 \times c_2^*, \\
K &= N_1 \times s^*, \\
L &= N_1, \\
Y &= A \times \left(K^\alpha + L^{1-\alpha}\right), \\
r &= \alpha A K^{\alpha-1}, \\
w &= \left(1-\alpha\right) A L^{-\alpha}.
\end{aligned}$$

You can solve this by substituting.

References

Williamson, Steven D. 2013. *Macroeconomics.* Addison-Wesley.